

On the Time Derivative in an Obstacle Problem

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ABSTRACT: We prove that the time derivative of the solution for the obstacle problem related to the Evolutionary p -Laplace Equation exists in Sobolev's sense, provided that the given obstacle is smooth enough. We keep $p \geq 2$.

1 Introduction

The celebrated *Evolutionary p -Laplace Equation* is much studied and the regularity theory for the solutions is almost complete. We refer to the book [dB] about this fascinating equation. In general, the corresponding subsolutions and supersolutions do not possess that much regularity, they are semicontinuous. We are interested in a special kind of weak supersolutions of the Evolutionary p -Laplace Equation, namely the solutions of an obstacle problem. In the presence of a smooth obstacle the regularity improves a lot. Given a function $\psi = \psi(x, t)$ in a bounded domain $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbf{R}^n$, we consider all functions v such that

$$\frac{\partial v}{\partial t} \geq \nabla \cdot (|\nabla v|^{p-2} \nabla v) \quad \text{and} \quad v \geq \psi \quad \text{in} \quad \Omega_T.$$

The function ψ acts as an *obstacle*. The smallest admissible v is the solution of the obstacle problem. (This makes sense because a comparison principle is valid.) However, the above description was only formal. We will instead use Definition 1 below, which is more adequate since it comes with a **variational inequality**. —We will restrict ourselves to the case $p > 2$, the so-called slow diffusion case.

It is an established fact that if the obstacle ψ is smooth enough, the solution to the obstacle problem inherits some regularity. Our objective is the time derivative u_t of the solution u , which *a priori* is only known to be a distribution. Our main result Theorem 2 states that, if ψ has continuous second derivatives, then the time derivative u_t exists in Sobolev's sense and it belongs to the space $L_{loc}^{p/(p-1)}(\Omega_T)$. A formula is given for the derivative. The most laborious part of the proof is to show that $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is a function so that the rule

$$\int_0^T \int_{\Omega} \varphi \Delta_p u \, dx \, dt = - \int_0^T \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, dx \, dt$$

with test functions applies. The equation has first to be regularized, keeping the obstacle unaffected, and then difference quotients are used. The test functions in [L1] can be adjusted to work here.

An important feature, typical for obstacle problems, is that in the open set $\Upsilon = \{u > \psi\}$ where the obstacle does not hinder, u is, actually, a solution to the differential equation. Thus in Υ the equation $u_t = \Delta_p u$ holds in the weak sense. The boundary of the *coincidence set* $\Xi = \{u = \psi\}$ is crucial. This enables us to get an identity for the integral $\iint u \varphi_t \, dx \, dt$, from which one can deduce the existence of the time derivative sought for. The special case with no obstacle present was treated in [L2]. —See also [BDM] for some general comments valid for “irregular” obstacles.

To this we may add a curious fact valid for $\psi \in C^2(\Omega_T)$. At all points in the coincidence set Ξ the obstacle satisfies the inequality

$$\frac{\partial \psi}{\partial t} \geq \Delta_p \psi.$$

Thus a point at which $\frac{\partial \psi}{\partial t} < \Delta_p \psi$ cannot belong to the coincidence set. This piece of information follows from the characterization of continuous supersolutions as *viscosity* supersolutions, cf. [JLM]. Then ψ itself can do as a test function for the pointwise testing required in the theory of viscosity solutions. (The reader may consult [K] for some basic concepts.) —We will not need this observation.

It is likely that the time derivative belongs to the space $L_{loc}^2(\Omega_T)$, but an eventual proof of this improvement would require much stronger regularity considerations for ∇u . We have kept $p > 2$, but one can expect a counterpart to Theorem 2 valid in the extended range $p > 2n/(n+2)$. The difficulty about further generalizations with $\Delta_p u$ replaced by some operator $\operatorname{div} \mathbf{A}_p$ is

the following. It is absolutely necessary that the solutions of the differential equation

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{A}_p(x, t, u, \nabla u)$$

enjoy the property of having a time derivative themselves, in order that the corresponding results could be extended to the related obstacle problem. This considerably restricts the possibilities.

2 Preliminaries

Let Ω be a bounded domain in the n -dimensional space \mathbf{R}^n having a Lipschitz regular boundary. Suppose that a function $\psi = \psi(x, t)$ is given in the closure of the space-time cylinder $\Omega_T = \Omega \times (0, T)$. The function ψ acts as an *obstacle* so that the admissible functions are forced to lie above ψ in Ω_T . We make the

$$\textbf{Assumption: } \psi \in C(\overline{\Omega_T}) \cap W^{2,p}(\Omega_T).$$

For simplicity the obstacle ψ also determines the values of the admissible functions on the *parabolic boundary*

$$\Gamma_T = \Omega \times \{0\} \cup \partial\Omega \times [0, T].$$

The *class of admissible functions* is

$$\mathcal{F}_\psi = \{v \in L^p(0, T; W^{1,p}(\Omega)) \mid v \in C(\overline{\Omega_T}), v \geq \psi \text{ in } \Omega_T, v = \psi \text{ on } \Gamma_T\}.$$

—We keep $p \geq 2$.

Definition 1 *We say that the function $u \in \mathcal{F}_\psi$ is the solution to the obstacle problem, if the inequality*

$$\begin{aligned} \int_0^T \int_\Omega \left(\langle |\nabla u|^{p-2} \nabla u, \nabla(\phi - u) \rangle + (\phi - u) \frac{\partial \phi}{\partial t} \right) dx dt \\ \geq \frac{1}{2} \int_\Omega (\phi(x, T) - u(x, T))^2 dx \end{aligned} \quad (1)$$

holds for all smooth functions $\phi \in \mathcal{F}_\psi$.

The solution exists and is unique, cf.[AL] and [C]. See also [KKS]. It is also a supersolution of the equation $u_t \geq \Delta_p u$, i.e.,

$$\int_0^T \int_{\Omega} \left(\langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle - u \frac{\partial \varphi}{\partial t} \right) dx dt \geq 0 \quad (2)$$

for all non-negative $\varphi \in C_0^\infty(\Omega_T)$.

Notice that nothing is assumed about the time derivative u_t . Our main result is the theorem below.

Theorem 2 *The time derivative u_t of the solution u to the obstacle problem exists in the Sobolev sense and $u_t \in L_{loc}^{p/(p-1)}(\Omega_T)$. It is the function*

$$u_t = \begin{cases} \psi_t & \text{in } \Xi \\ \Delta_p u & \text{in } \Omega_T \setminus \Xi \end{cases}$$

where $\Xi = \{u = \psi\}$ denotes the coincidence set.

In order to avoid the difficulty with the “forbidden” time derivative u_t in the proof, we have to regularize the equation, keeping the obstacle unchanged. We replace $|\nabla u|^{p-2} \nabla u$ by

$$\left(|\nabla u|^2 + \varepsilon^2 \right)^{\frac{p-2}{2}} \nabla u$$

to obtain an equation which does not degenerate as $\nabla u = 0$.

Lemma 3 *There is a unique $u^\varepsilon \in \mathcal{F}_\psi$ such that*

$$\begin{aligned} & \int_0^T \int_{\Omega} \left(\langle |\nabla u^\varepsilon|^2 + \varepsilon^2 \rangle^{\frac{p-2}{2}} \nabla u^\varepsilon, \nabla(\phi - u^\varepsilon) \rangle + (\phi - u^\varepsilon) \frac{\partial \phi}{\partial t} \right) dx dt \\ & \geq \frac{1}{2} \int_{\Omega} (\phi(x, T) - u^\varepsilon(x, T))^2 dx \end{aligned} \quad (3)$$

for all smooth functions ϕ in the class \mathcal{F}_ψ . In the open set $\{u^\varepsilon > \psi\}$ the function u^ε is a solution of the equation

$$u_t^\varepsilon = \nabla \cdot \left(\left(|\nabla u^\varepsilon|^2 + \varepsilon^2 \right)^{\frac{p-2}{2}} \nabla u^\varepsilon \right).$$

In the case $\varepsilon \neq 0$ we have $u^\varepsilon \in C^\infty(\Omega_T)$ and $\frac{\partial u^\varepsilon}{\partial t} \in L^2(\Omega_T)$.

Proof: The existence can be extracted from the proof of [AL, Theorem 3.2]. The regularity for the nondegenerate case $\varepsilon \neq 0$ is according to the standard parabolic theory described in the celebrated book [LSU]. The proof of the Hölder continuity for the degenerate case $\varepsilon = 0$ is in [C].

When $\varepsilon \neq 0$, we can rewrite equation (3) in the more convenient form

$$\int_0^T \int_{\Omega} \underbrace{\left(|\nabla u^\varepsilon|^2 + \varepsilon^2 \right)^{\frac{p-2}{2}} \nabla u^\varepsilon, \nabla \eta}_{\mathbf{A}_\varepsilon(x,t)} + \eta \frac{\partial u^\varepsilon}{\partial t} dx dt \geq 0 \quad (4)$$

valid for all test functions η such that $\eta \geq \psi - u^\varepsilon$ in Ω_T and $\eta = 0$ on Γ_T . We may even use any continuous $\eta \in L^p(0, T; W_0^{1,p}(\Omega))$ with $\eta(x, 0) = 0$.

In order to proceed to the limit under the integral sign in the forthcoming equations we need the convergence result below, where u denotes the solution to the original obstacle problem, the one with $\varepsilon = 0$.

Lemma 4

$$\lim_{k \rightarrow 0} \int_0^T \int_{\Omega} \left(|u^\varepsilon - u|^p + |\nabla u^\varepsilon - \nabla u|^p \right) dx dt = 0. \quad (5)$$

Proof: It was established in [KL, Lemma 3.2] that

$$\lim_{k \rightarrow 0} \int_0^T \int_{\Omega} |\nabla u^\varepsilon - \nabla u|^p dx dt = 0, \quad (6)$$

but the strong convergence of the functions themselves requires, as it were, an extra compactness argument. Since u^ε is a weak supersolution, there exists a Radon measure μ_ε such that

$$\int_0^T \int_{\Omega} \left(\left(|\nabla u^\varepsilon|^2 + \varepsilon^2 \right)^{\frac{p-2}{2}} \nabla u^\varepsilon, \nabla \varphi \right) - u^\varepsilon \frac{\partial \varphi}{\partial t} dx dt = \int_{\Omega_T} \varphi d\mu_\varepsilon$$

for all functions $\varphi \in C_0^\infty(\Omega_T)$, whether positive or not. This is a consequence of Riesz's Representation Theorem, cf. [EG, 1.8]. See [KLP] for details.

Given a regular open set (for example a polyhedron) $U \subset\subset \Omega_T$, we have to verify that

$$\mu_\varepsilon(U) \leq M_U$$

with a bound independent of ε , $0 < \varepsilon < 1$. Then the lemma follows as in [KLP, pp. 720-721]. (There [S] was used.) To this end, choose a test function $\varphi \in C_0^\infty(\Omega_T)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ in U . A rough estimation yields

$$\begin{aligned}\mu_\varepsilon(U) &= \int_{\Omega_T} d\mu_\varepsilon = \int_{\Omega_T} \varphi d\mu_\varepsilon \\ &= \int_0^T \int_\Omega \left(\langle |\nabla u^\varepsilon|^2 + \varepsilon^2 \rangle^{\frac{p-2}{2}} \nabla u^\varepsilon, \nabla \varphi \rangle - u^\varepsilon \frac{\partial \varphi}{\partial t} \right) dx dt \\ &\leq C_1 \left(\|\nabla u^\varepsilon\|_{L^p(\Omega_T)}^p + \varepsilon^{\frac{p(p-2)}{p-1}} \right) + C_2 \|u^\varepsilon\|_\infty.\end{aligned}$$

By the maximum principle $\|u^\varepsilon\|_\infty \leq \|\psi\|_\infty$ and $\|\nabla u^\varepsilon\|_{L^p(\Omega_T)}^p$ is uniformly bounded, since the gradients converge strongly. This yields the bound M_U . \square

3 The gradient estimate

In order to prove that $\Delta_p u$ is a *function*, u denoting the solution to the obstacle problem, we show that the function $\mathbf{F} = |\nabla u|^{(p-2)/2} \nabla u$, where the usual power $p-2$ has been replaced by $(p-2)/2$, is in a suitable first order Sobolev x -space. This will immediately imply the desired result. At a first reading one had better to assume that the obstacle ψ is as smooth as one pleases, say of class $C^2(\overline{\Omega_T})$. Actually, only the Sobolev derivatives $\psi_{x_i x_j}$ and $\psi_{x_i t}$ are needed, while ψ_{tt} does not appear at all. We recall our assumption $\psi \in C(\overline{\Omega_T}) \cap W^{2,p}(\Omega_T)$ and use the abbreviation

$$|D^2 \psi|^2 = \sum \psi_{x_i x_j}^2.$$

Under these assumptions about the obstacle $\psi = \psi(x, t)$ we have the following result.

Theorem 5 *For the solution u to the obstacle problem, the derivative $D\mathbf{F}$ of*

$$\mathbf{F} = |\nabla u|^{\frac{(p-2)}{2}} \nabla u$$

exists in Sobolev's sense and belongs to $L_{loc}^{p/(p-1)}(\Omega_T)$. The estimate

$$\begin{aligned} \int_0^T \int_{\Omega} \zeta^p |D\mathbf{F}|^2 dx dt &\leq C \int_0^T \int_{\Omega} (\zeta^p + |\nabla \zeta|^p) |\nabla u|^p dx dt \\ &+ C \int_0^T \int_{\Omega} \zeta^p |\nabla u|^2 dx dt + C \int_0^T \int_{\Omega} |\nabla \zeta|^p |\nabla \psi|^p dx dt \\ &+ C \int_0^T \int_{\Omega} \zeta^p (|D^2 \psi|^p + |\nabla \psi_t|^2) dx dt + C \int_{\Omega} \zeta^p |\nabla \psi(x, T)|^2 dx \end{aligned}$$

holds for each non-negative test function $\zeta = \zeta(x)$ in $C_0^\infty(\Omega)$; and $C = C(p)$.

Proof: The proof is based on the regularized obstacle problem and equation (4), where we abbreviate

$$\mathbf{A}_\varepsilon(x, t) = (|\nabla u^\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \nabla u^\varepsilon.$$

We denote its solution by u , suppressing the index ε . Thus u means u^ε , to begin with. Given ζ , the variable x is given a small increment h so that the test function

$$\begin{aligned} \eta &= \psi(x, t) - u(x, t) + \zeta(x)^p [u(x+h, t) - \psi(x+h, t)] \\ &= \zeta(x)^p \overbrace{[u(x+h, t) - u(x, t)]}^{\Delta_{\mathbf{h}} u} - \zeta(x)^p \overbrace{[\psi(x+h, t) - \psi(x, t)]}^{\Delta_{\mathbf{h}} \psi} \\ &\quad - (1 - \zeta(x)^p) [u(x, t) - \psi(x, t)] \end{aligned}$$

is admissible in the regularized equation

$$\int_0^T \int_{\Omega} \left(\langle \mathbf{A}_\varepsilon(x, t), \nabla \eta \rangle + \eta \frac{\partial u}{\partial t} \right) dx dt \geq 0. \quad (7)$$

Inserting the test function, we obtain

$$\begin{aligned} &\int_0^T \int_{\Omega} \left(\langle \mathbf{A}_\varepsilon(x, t), \nabla (\zeta^p \Delta_{\mathbf{h}} u) \rangle + \zeta^p \Delta_{\mathbf{h}} u \frac{\partial u}{\partial t} \right) dx dt \\ &- \int_0^T \int_{\Omega} \left(\langle \mathbf{A}_\varepsilon(x, t), \nabla (\zeta^p \Delta_{\mathbf{h}} \psi) \rangle + \zeta^p \Delta_{\mathbf{h}} \psi \frac{\partial u}{\partial t} \right) dx dt \\ &\geq \int_0^T \int_{\Omega} \left(\langle \mathbf{A}_\varepsilon(x, t), \nabla ((1 - \zeta(x)^p) [u(x, t) - \psi(x, t)]) \rangle \right. \\ &\quad \left. + (1 - \zeta(x)^p) [u(x, t) - \psi(x, t)] \frac{\partial u}{\partial t} \right) dx dt \\ &\geq 0. \end{aligned}$$

The last integral is non-negative, because

$$(1 - \zeta(x)^p)[u(x, t) - \psi(x, t)]$$

will do as a test function in the equation (7). This observation is important here.

Aiming at difference quotients we give x the increment h . The translated function $u(x + h, t)$ solves the obstacle problem with the translated obstacle $\psi(x + h, t)$, all this with respect to the shifted domain $\Omega^h \times (0, T)$ where $\Omega^h = \{x \mid x + h \in \Omega\}$. For sufficiently small h we have

$$\int_0^T \int_{\Omega^h} \left(\langle \mathbf{A}_\varepsilon(x + h, t), \nabla \eta(x, t) \rangle + \eta(x, t) \frac{\partial u(x + h, t)}{\partial t} \right) dx dt \geq 0 \quad (8)$$

whenever $\eta(x, t) \geq \psi(x + h, t) - u(x + h, t)$ and $\eta = 0$ on the parabolic boundary of $\Omega^h \times (0, T)$. Here

$$\begin{aligned} \eta &= \psi(x + h, t) - u(x + h, t) + \zeta(x)^p[u(x, t) - \psi(x, t)] \\ &= \zeta(x)^p \overbrace{[u(x + h, t) - u(x, t)]}^{\Delta_h u} - \zeta^p(x) \overbrace{[\psi(x + h, t) - \psi(x, t)]}^{\Delta_h \psi} \\ &\quad - (1 - \zeta(x)^p)[u(x + h, t) - \psi(x + h, t)] \end{aligned}$$

will do. We obtain

$$\begin{aligned} & - \int_0^T \int_{\Omega^h} \left(\langle \mathbf{A}_\varepsilon(x + h, t), \nabla(\zeta^p \Delta_h u) \rangle + \zeta^p \Delta_h u \frac{\partial u(x + h, t)}{\partial t} \right) dx dt \\ & + \int_0^T \int_{\Omega^h} \left(\langle \mathbf{A}_\varepsilon(x + h, t), \nabla(\zeta^p \Delta_h \psi) \rangle + \zeta^p \Delta_h \psi \frac{\partial u(x + h, t)}{\partial t} \right) dx dt \\ & \geq \int_0^T \int_{\Omega^h} \left(\langle \mathbf{A}_\varepsilon(x + h, t), \nabla((1 - \zeta(x)^p)[u(x + h, t) - \psi(x + h, t)]) \rangle \right. \\ & \quad \left. + (1 - \zeta(x)^p)[u(x + h, t) - \psi(x + h, t)] \frac{\partial u(x + h, t)}{\partial t} \right) dx dt \\ & \geq 0. \end{aligned}$$

The last integral is positive because

$$(1 - \zeta(x)^p)[u(x + h, t) - \psi(x + h, t)]$$

will do as a test function in the translated equation (8). This observation is essential here. The integrals in the left-hand member of the inequality

are, in fact, taken only over the support of the function $\zeta(x)$. Hence we have an inequality with integrals taken only over Ω_T , provided that $|h| < \text{dist}(\text{supp}\zeta, \partial\Omega)$. Thus Ω^h is no longer directly involved.

We add the two estimates, grouping the differences, and obtain

$$\begin{aligned} & + \int_0^T \int_{\Omega} \langle \mathbf{A}_{\varepsilon}(x+h, t) - \mathbf{A}_{\varepsilon}(x, t), \nabla(\zeta^p \Delta_{\mathbf{h}} u) \rangle dx dt \\ & \leq \int_0^T \int_{\Omega} \langle \mathbf{A}_{\varepsilon}(x+h, t) - \mathbf{A}_{\varepsilon}(x, t), \nabla(\zeta^p \Delta_{\mathbf{h}} \psi) \rangle dx dt \\ & - \int_0^T \int_{\Omega} \zeta^p \Delta_{\mathbf{h}} u \cdot \Delta_{\mathbf{h}} \left(\frac{\partial u}{\partial t} \right) dx dt + \int_0^T \int_{\Omega} \zeta^p \Delta_{\mathbf{h}} \psi \cdot \Delta_{\mathbf{h}} \left(\frac{\partial u}{\partial t} \right) dx dt. \end{aligned}$$

The integrals with the time derivatives can be integrated by parts:

$$\begin{aligned} & - \int_0^T \int_{\Omega} \zeta^p \frac{\partial}{\partial t} \frac{(\Delta_{\mathbf{h}} u)^2}{2} dx dt + \int_0^T \int_{\Omega} \zeta^p \Delta_{\mathbf{h}} \psi \cdot \Delta_{\mathbf{h}} \left(\frac{\partial u}{\partial t} \right) dx dt \\ & = - \int_{\Omega} \zeta^p(x) \frac{(\Delta_{\mathbf{h}} u)^2}{2} \Big|_0^T dx + \int_{\Omega} \zeta^p(x) \Delta_{\mathbf{h}} \psi \cdot \Delta_{\mathbf{h}} u \Big|_0^T dx \\ & - \int_0^T \int_{\Omega} \zeta^p \Delta_{\mathbf{h}} u \cdot \Delta_{\mathbf{h}} \left(\frac{\partial \psi}{\partial t} \right) dx dt. \end{aligned}$$

Since $\Delta_{\mathbf{h}} u = \Delta_{\mathbf{h}} \psi$ when $t = 0$, the above expression is majorized by

$$\frac{1}{2} \int_{\Omega} \zeta^p ((\Delta_{\mathbf{h}} \psi)_T^2 - (\Delta_{\mathbf{h}} \psi)_0^2) dx + \frac{1}{2} \int_0^T \int_{\Omega} \zeta^p \left((\Delta_{\mathbf{h}} u)^2 + \left(\Delta_{\mathbf{h}} \frac{\partial \psi}{\partial t} \right)^2 \right) dx dt,$$

where the inequality $2\Delta_{\mathbf{h}} u \Delta_{\mathbf{h}} \psi \leq (\Delta_{\mathbf{h}} u)^2 + (\Delta_{\mathbf{h}} \psi)^2$ was used at time T .

At this stage there are no “forbidden” time derivatives left and so we may safely let ε go to zero. By Lemma 3 we may pass to the limit under the integral sign and hence the estimate for the limit u (no longer u^{ε}) becomes

$$\begin{aligned} & \int_0^T \int_{\Omega} (\langle \Delta_{\mathbf{h}} \mathbf{A}, \nabla(\zeta^p \Delta_{\mathbf{h}} u) \rangle) dx dt \\ & \leq \int_0^T \int_{\Omega} (\langle \Delta_{\mathbf{h}} \mathbf{A}, \nabla(\zeta^p \Delta_{\mathbf{h}} \psi) \rangle) dx dt \\ & + \frac{1}{2} \int_0^T \int_{\Omega} \zeta^p \left((\Delta_{\mathbf{h}} u)^2 + \left(\Delta_{\mathbf{h}} \frac{\partial \psi}{\partial t} \right)^2 \right) dx dt + \frac{1}{2} \int_{\Omega} \zeta^p (\Delta_{\mathbf{h}} \psi)_T^2 dx, \end{aligned}$$

where

$$\Delta_{\mathbf{h}}\mathbf{A} = \mathbf{A}(x+h, t) - \mathbf{A}(x, t).$$

We write this more conveniently as

$$\begin{aligned} & \int_0^T \int_{\Omega} \zeta^p \langle \Delta_{\mathbf{h}}\mathbf{A}, \nabla \Delta_{\mathbf{h}}u \rangle dx dt \\ & \leq \overbrace{\int_0^T \int_{\Omega} p \zeta^{p-1} |\Delta_{\mathbf{h}}\mathbf{A}| |\Delta_{\mathbf{h}}u| |\nabla \zeta| dx dt}^{\text{I}} \\ & + \overbrace{\int_0^T \int_{\Omega} p \zeta^{p-1} |\Delta_{\mathbf{h}}\mathbf{A}| |\Delta_{\mathbf{h}}\psi| |\nabla \zeta| dx dt}^{\text{II}} \\ & + \overbrace{\int_0^T \int_{\Omega} \zeta^p |\Delta_{\mathbf{h}}\mathbf{A}| |\nabla \Delta_{\mathbf{h}}\psi| dx dt}^{\text{III}} \\ & + \frac{1}{2} \int_0^T \int_{\Omega} \zeta^p \left((\Delta_{\mathbf{h}}u)^2 \left(\Delta_{\mathbf{h}} \frac{\partial \psi}{\partial t} \right)^2 \right) dx dt + \frac{1}{2} \int_{\Omega} \zeta^p (\Delta_{\mathbf{h}}\psi)_T^2 dx. \end{aligned} \tag{9}$$

The integrand on left-hand side is $\langle \Delta_{\mathbf{h}}\mathbf{A}, \nabla \Delta_{\mathbf{h}}u \rangle =$

$$\begin{aligned} & \langle |\nabla u(x+h, t)|^{\frac{p-2}{2}} \nabla u(x+h, t) - |\nabla u(x, t)|^{\frac{p-2}{2}} \nabla u(x, t), \nabla u(x+h, t) - \nabla u(x, t) \rangle \\ & \geq \frac{4}{p^2} |\mathbf{F}(x+h, t) - \mathbf{F}(x, t)|^2 = \frac{4}{p^2} |\Delta_{\mathbf{h}}\mathbf{F}|^2, \end{aligned} \tag{10}$$

where the elementary inequality

$$\frac{4}{p^2} \left| |b|^{\frac{p-2}{2}} b - |a|^{\frac{p-2}{2}} a \right|^2 \leq \langle |b|^{p-2} b - |a|^{p-2} a, b - a \rangle$$

for vectors was used. We aim at an estimate for the integral of $\zeta^p |\Delta_{\mathbf{h}}\mathbf{F}|$.

We divide the $\Delta_{\mathbf{h}}$ -terms by $|h|$ so that the desired difference quotients appear. The estimate

$$\left| \frac{\Delta_{\mathbf{h}}\mathbf{A}}{h} \right| \leq (p-1) \left| \frac{\Delta_{\mathbf{h}}\mathbf{F}}{h} \right| (|\nabla u(x+h, t)|^{\frac{p-2}{2}} + |\nabla u(x, t)|^{\frac{p-2}{2}}),$$

coming from the elementary vector inequality

$$||b|^{p-2} b - |a|^{p-2} a| \leq (p-1) \left(|b|^{\frac{p-2}{2}} + |a|^{\frac{p-2}{2}} \right) \left| |b|^{\frac{p-2}{2}} b - |a|^{\frac{p-2}{2}} a \right|,$$

is used in the integrands of I, II, and III. In I we split the factors so that

$$\begin{aligned}
& p\zeta^{p-1} \left| \frac{\Delta_{\mathbf{h}} \mathbf{A}}{h} \right| \left| \frac{\Delta_{\mathbf{h}} u}{h} \right| |\nabla \zeta| \\
& \leq p(p-1) \left[\zeta^{\frac{p}{2}} \left| \frac{\Delta_{\mathbf{h}} \mathbf{F}}{h} \right| \right] \left[\left| \frac{\Delta_{\mathbf{h}} u}{h} \right| |\nabla \zeta| \right] \left[\zeta^{\frac{p-2}{2}} (|\nabla u(x, t)|^{\frac{p-2}{2}} + |\nabla u(x+h, t)|^{\frac{p-2}{2}}) \right]
\end{aligned}$$

and use Young's inequality

$$abc \leq \frac{\varepsilon^2 a^2}{2} + \frac{\varepsilon^{-p} b^p}{p} + \frac{(p-2)c^{\frac{2p}{p-2}}}{2p}$$

to get the bound

$$\begin{aligned}
\frac{\text{I}}{|h|^2} & \leq \frac{p(p-1)\varepsilon^2}{2} \int_0^T \int_{\Omega} \zeta^p \left| \frac{\Delta_{\mathbf{h}} \mathbf{F}}{h} \right|^2 dx dt \\
& \quad + (p-1)\varepsilon^{-p} \int_0^T \int_{\Omega} \left| \frac{\Delta_{\mathbf{h}} u}{h} \right|^p |\nabla \zeta|^p dx dt \\
& \quad + c_p \int_0^T \int_{\Omega} \zeta^p (|\nabla u(x, t)|^p + |\nabla u(x+h, t)|^p) dx dt
\end{aligned}$$

The integral II/ $|h|^2$ has a similar majorant, the only difference being that $\Delta_{\mathbf{h}} u$ be replaced by $\Delta_{\mathbf{h}} \psi$. The integrand of III is estimated in a similar way:

$$\begin{aligned}
& p\zeta^p \left| \frac{\Delta_{\mathbf{h}} \mathbf{A}}{h} \right| \left| \frac{\Delta_{\mathbf{h}} \psi}{h} \right| \\
& \leq p(p-1) \left[\zeta^{\frac{p}{2}} \left| \frac{\Delta_{\mathbf{h}} \mathbf{F}}{h} \right| \right] \left[\zeta \left| \nabla \left(\frac{\Delta_{\mathbf{h}} u}{h} \right) \right| \right] \left[\zeta^{\frac{p-2}{2}} (|\nabla u(x, t)|^{\frac{p-2}{2}} + |\nabla u(x+h, t)|^{\frac{p-2}{2}}) \right] \\
& \leq \frac{p(p-1)\varepsilon^2}{2} \zeta^p \left| \frac{\Delta_{\mathbf{h}} \mathbf{F}}{h} \right|^2 + (p-1)\varepsilon^{-p} \zeta^p \left| \nabla \left(\frac{\Delta_{\mathbf{h}} \psi}{h} \right) \right|^p \\
& \quad + c_p \zeta^p (|\nabla u(x, t)|^p + |\nabla u(x+h, t)|^p).
\end{aligned}$$

Adding up the three integrated estimates, we arrive at

$$\begin{aligned}
& \frac{\text{I} + \text{II} + \text{III}}{|h|^2} \leq \\
& 3 \frac{p(p-1)\varepsilon^2}{2} \int_0^T \int_{\Omega} \zeta^p \left| \frac{\Delta_{\mathbf{h}} \mathbf{F}}{h} \right|^2 dx dt \\
& + (p-1)\varepsilon^{-p} \int_0^T \int_{\Omega} \left(\left| \frac{\Delta_{\mathbf{h}} u}{h} \right|^p |\nabla \zeta|^p + \left| \frac{\Delta_{\mathbf{h}} \psi}{h} \right|^p |\nabla \zeta|^p + \zeta^p \left| \frac{\nabla(\Delta_{\mathbf{h}} \psi)}{h} \right|^p \right) dx dt \\
& + 3c_p \int_0^T \int_{\Omega} \zeta^p \left(|\nabla u(x, t)|^p + |\nabla u(x+h, t)|^p \right) dx dt.
\end{aligned}$$

This complements (9). Recall (10). The next step is to absorb the first integral above in the right-hand member into the minorant in (10) by fixing ε small enough, say

$$3 \frac{p(p-1)\varepsilon^2}{2} = \frac{2}{p^2}.$$

The resulting estimate, written out without abbreviations, is

$$\begin{aligned}
& \int_0^T \int_{\Omega} \zeta^p \left| \frac{\mathbf{F}(x+h, t) - \mathbf{F}(x, t)}{h} \right|^2 dx dt \\
& \leq a_p \int_0^T \int_{\Omega} \left| \frac{u(x+h, t) - u(x, t)}{h} \right|^p |\nabla \zeta|^p dx dt \\
& + a_p \int_0^T \int_{\Omega} \left| \frac{\psi(x+h, t) - \psi(x, t)}{h} \right|^p |\nabla \zeta|^p dx dt \\
& + a_p \int_0^T \int_{\Omega} \zeta^p \left| \frac{\nabla \psi(x+h, t) - \nabla \psi(x, t)}{h} \right|^p dx dt \\
& + b_p \int_0^T \int_{\Omega} \zeta^p \left(|\nabla u(x, t)|^p + |\nabla u(x+h, t)|^p \right) dx dt \\
& + c_p \int_0^T \int_{\Omega} \zeta^p \left| \frac{u(x+h, t) - u(x, t)}{h} \right|^2 dx dt \\
& + c_p \int_0^T \int_{\Omega} \zeta^p \left| \frac{\psi_t(x+h, t) - \psi_t(x, t)}{h} \right|^2 dx dt \\
& + c_p \int_{\Omega} \zeta^p \left| \frac{\psi(x+h, T) - \psi(x, T)}{h} \right|^2 dx,
\end{aligned}$$

where the constants depend only on p . Finally, letting the increment $h \rightarrow 0$ in any desired direction, we arrive at the estimate in the theorem. Here we use the characterization of Sobolev spaces in terms of integrated differential quotients, cf. [G, Chapter 8.1]. This concludes our proof of Theorem 5.

Corollary 6 *If u is the solution to the obstacle problem with the obstacle ψ , then $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ belongs to the space $L_{loc}^{\frac{p}{p-1}}(\Omega_T)$ and*

$$\int_0^T \int_{\Omega} \varphi \Delta_p u \, dx \, dt = \int_0^T \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, dx \, dt$$

for all test functions φ in $C_0^\infty(\Omega_T)$.

Proof: Since \mathbf{F} is in Sobolev's space and $p \geq 2$, we can differentiate

$$|\nabla u|^{p-2} \nabla u = |\mathbf{F}|^{\frac{p-2}{p}} \mathbf{F}$$

and hence

$$\left| \frac{\partial}{\partial x_j} (|\nabla u|^{p-2} \nabla u) \right| \leq 2 \left(1 - \frac{1}{p} \right) |\mathbf{F}|^{\frac{p-2}{p}} \left| \frac{\partial \mathbf{F}}{\partial x_j} \right|.$$

By Hölder's inequality

$$\frac{\partial}{\partial x_j} (|\nabla u|^{p-2} \nabla u) \in L_{loc}^{\frac{p}{p-1}}(\Omega_T),$$

since $\mathbf{F} \in L^2(\Omega_T)$ and $D\mathbf{F} \in L^2(\Omega_T)$. \square

4 The Time Derivative

For the proof of the Theorem we notice that the contact set $\Xi = \{u = \psi\}$ is a closed subset of $\overline{\Omega_T}$ and that its complement $\Upsilon = \Omega_T \setminus \Xi$ is open. In the set Υ , where the obstacle does not hinder, u is a solution to the Evolutionary p -Laplace Equation $u_t = \Delta_p u$. In other words, whenever $\phi \in C_0^\infty(\Upsilon)$,

$$\int_0^T \int_{\Omega} u \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^T \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, dx \, dt = - \int_0^T \int_{\Omega} \phi \Delta_p u \, dx \, dt,$$

the actual set of integration being Υ . Here Corollary was used. Thus u_t is available, but only in Υ to begin with. (See also [L2].) Let ϕ denote an

arbitrary test function in $C_0^\infty(\Omega_T)$. We need a specific test function with compact support in Υ . To construct it, define

$$\theta_k = \min\{1, k(u - \psi)\}, \quad k = 1, 2, \dots$$

Then $1 - \theta_k = 1$ in Ξ and pointwise the monotone convergence $1 - \theta_k \rightarrow \chi_\Xi$ holds. Moreover, the support of θ_k is compact in Υ . The time derivative of θ_k is available!

Using

$$\int_0^T \int_\Omega u \frac{\partial}{\partial t} (\theta_k \phi) dx dt = \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla (\theta_k \phi) \rangle dx dt,$$

we write

$$\begin{aligned} & \int_0^T \int_\Omega \phi \Delta_p u dx dt = - \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle dx dt \\ &= - \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla (\theta_k \phi + (1 - \theta_k) \phi) \rangle dx dt \\ &= - \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla (\theta_k \phi) \rangle dx dt - \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla ((1 - \theta_k) \phi) \rangle dx dt \\ &= - \int_0^T \int_\Omega u \frac{\partial}{\partial t} (\theta_k \phi) dx dt + \int_0^T \int_\Omega (1 - \theta_k) \phi \Delta_p u dx dt. \end{aligned}$$

The last integral has the limit

$$\lim_{k \rightarrow 0} \int_0^T \int_\Omega (1 - \theta_k) \phi \Delta_p u dx dt = \int_\Xi \int \phi \Delta_p \psi dx dt.$$

In the integral with the time derivative we write

$$-u \frac{\partial}{\partial t} (\theta_k \phi) = -u \frac{\partial \phi}{\partial t} + (u - \psi) \frac{\partial}{\partial t} (1 - \theta_k) \phi + \psi \frac{\partial}{\partial t} (1 - \theta_k) \phi$$

and obtain

$$\begin{aligned} & - \int_0^T \int_\Omega u \frac{\partial}{\partial t} (\theta_k \phi) dx dt = \int_0^T \int_\Omega -u \frac{\partial \phi}{\partial t} dx dt \\ & + \int_0^T \int_\Omega (u - \psi) \frac{\partial}{\partial t} ((1 - \theta_k) \phi) dx dt - \int_0^T \int_\Omega (1 - \theta_k) \phi \frac{\partial \psi}{\partial t} dx dt, \end{aligned}$$

where an integration by parts has produced the last integral. It has the evident limit

$$\lim_{k \rightarrow 0} \int_0^T \int_{\Omega} (1 - \theta_k) \phi \frac{\partial \psi}{\partial t} dx dt = \int \int_{\Xi} \phi \frac{\partial \psi}{\partial t} dx dt.$$

The middle integral vanishes as $k \rightarrow 0$:

$$\begin{aligned} & \int_0^T \int_{\Omega} (u - \psi) \frac{\partial}{\partial t} ((1 - \theta_k) \phi) dx dt \\ &= \int_0^T \int_{\Omega} (u - \psi) (1 - \theta_k) \frac{\partial \phi}{\partial t} dx dt - \int_0^T \int_{\Omega} \phi (u - \psi) \frac{\partial \theta_k}{\partial t} dx dt \\ &= \int_0^T \int_{\Omega} (u - \psi) (1 - \theta_k) \frac{\partial \phi}{\partial t} dx dt - \frac{1}{2k} \int_0^T \int_{\Omega} \phi \frac{\partial}{\partial t} \theta_k^2 dx dt \\ &= \int_0^T \int_{\Omega} (u - \psi) (1 - \theta_k) \frac{\partial \phi}{\partial t} dx dt + \frac{1}{2k} \int_0^T \int_{\Omega} \theta_k^2 \frac{\partial \phi}{\partial t} dx dt \xrightarrow{k \rightarrow \infty} 0 + 0. \end{aligned}$$

Collecting results,

$$\int_0^T \int_{\Omega} \phi \Delta_p u dx dt = - \int_0^T \int_{\Omega} \phi_t dx dt - \int \int_{\Xi} (\psi_t - \Delta_p \psi) \phi dx dt.$$

In other words, the final formula

$$- \int_0^T \int_{\Omega} u \phi_t dx dt = \int_0^T \int_{\Omega} \phi [\Delta_p u + (\psi_t - \Delta_p \psi) \chi_{\Xi}] dx dt$$

holds for every ϕ in $C_0^\infty(\Omega_T)$. Therefore

$$u_t = \Delta_p u + (\psi_t - \Delta_p \psi) \chi_{\Xi}$$

and this is a *function* belonging to $L_{loc}^{p/(p-1)}(\Omega_T)$. This concludes the proof of Theorem 2. \square

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References

- [AL] H. Alt, S. Luckhaus. *Quasilinear elliptic-parabolic differential equations*, Mathematische Zeitschrift **183**, 1983, pp. 311–341.
- [BDM] V. Bögelein, F. Duzaar, G. Mingione. *Degenerate problems with irregular obstacles*, 2009. -To appear.
- [C] H.-J. Choe. *A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities*, Differential and Integral Equations **5**, 1992, pp. 915–944.
- [CIL] M. Crandall, H. Ishii, P.-L. Lions. *User’s guide to viscosity solutions of second order partial differential equations*, Bulletin of the American Mathematical Society **27**, 1992, pp. 1–67.
- [dB] E. DiBenedetto. *Degenerate Parabolic Equations*, Springer-Verlag, Berlin 1993.
- [EG] L. Evans, R. Gariepy. *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton 1992.
- [G] E. Giusti. *Metodi diretti nel calcolo delle variazioni*, Unione Matematica Italiana, Bologna 1994.
- [G] E. Giusti. *Direct Methods in the Calculus of Variations*, World Scientific, Singapore 2003.
- [JLM] P. Juutinen, P. Lindqvist, J. Manfredi. *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation*, SIAM Journal on Mathematical Analysis **33**, 2001, pp. 699–717.
- [KKS] R. Korte, T. Kuusi, J. Siljander. *Obstacle problems for nonlinear parabolic equations*, Journal of Differential Equations **246**, 2009, pp. 3668–3680.
- [KL] J. Kinnunen, P. Lindqvist. *Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation*, Annali di Matematica Pura ed Applicata (4), **185**, 2006, pp. 411–435.
- [KLP] J. Kinnunen, T. Lukkari, M. Parviainen. *An existence result for superparabolic functions*, Journal of Functional Analysis **258**, 2010, pp. 713–728.
- [Ko] S. Koike. *A Beginner’s Guide to the Theory of Viscosity Solutions*. (MSJ Memoirs **13**, Mathematical Society of Japan), Tokyo 2004.

- [LSU] O. Ladyzhenskaya, V. Solonnikov, N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence RI 1968.
- [L1] P. Lindqvist, *Regularity for the gradient of the solution to a nonlinear obstacle problem with degenerate ellipticity*, *Nonlinear Analysis, Theory, Methods & Applications* **12**, 1988, pp. 1245–1255.
- [L2] P. Lindqvist, *On the time derivative in a quasilinear equation*, *Transactions of the Royal Norwegian Society of Sciences and Letters* 2008 no. 2, pp. 1-7.
- [S] J. Simon, *Compact sets in the space $L^p(0, T; B)$* , *Annali di Matematica Pura ed Applicata* (4), **146**, 1987, pp. 65–96.